Example 4.29 ⟹ Determine the Z transform and ROC of signal given below using linearity property

$$x(n) = [5(2^n) - 4(3^n)]\,u(n)$$

Soln →

$$x_1(n) = 2^n u(n)$$

$$x_2(n) = 3^n u(n)$$

$$x(n) = 5x_1(n) - 4x_2(n)$$

According to linearity property

$$X(Z) = 5X_1(Z) - 4X_2(Z)$$

We recall that $\gamma^n u(n) = \dfrac{1}{1-\gamma Z^{-1}}$

$$x_1(n) = 2^n u(n) \longrightarrow X_1(Z) = \frac{1}{1-2Z^{-1}} \quad ROC\ |Z|>2$$

$$x_2(n) = 3^n u(n) \longrightarrow X_2(Z) = \frac{1}{1-3Z^{-1}} \quad ROC\ |Z|>3$$

The combined ROC is $|Z|>3$

$$X(Z) = \frac{5}{1-2Z^{-1}} - \frac{4}{1-3Z^{-1}} \quad ROC\ |Z|>3$$

Example 4.30 Determine the Z transform of following signal using linearity property

(i) $x(n) = (\cos\omega_0 n)\,u(n)$

(b) $x(n) = (\sin \omega_0 n)\, u(n)$

Soln We know that

$$\sin \omega_0 n = \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right]$$

$$\cos \omega_0 n = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right]$$

$$x(n) = \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) u(n)$$

$$= \underbrace{\frac{1}{2} e^{j\omega_0 n} u(n)}_{x_1(n)} + \underbrace{\frac{1}{2} e^{-j\omega_0 n} u(n)}_{x_2(n)}$$

$$x(n) = x_1(n) + x_2(n)$$

$$x_1(n) = \frac{1}{2} e^{j\omega_0 n} u(n) \overset{Z}{\longleftrightarrow} \frac{1}{2} \cdot \frac{1}{1 - e^{j\omega_0} z^{-1}} \quad \text{ROC } |z|>$$

$$x_2(n) = \frac{1}{2} e^{-j\omega_0 n} u(n) \overset{Z}{\longleftrightarrow} \frac{1}{2} \frac{1}{1 - e^{-j\omega_0} z^{-1}} \quad \text{ROC } |z|>1$$

$$X(z) = \frac{1}{2}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}} + \frac{1}{1 - e^{-j\omega_0} z^{-1}}\right] \quad \text{ROC } |z|>1$$

$$X(z) = \frac{1}{2}\left[\frac{1 - e^{-j\omega_0} z^{-1} + 1 - e^{j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}\right]$$

$$X(Z) = \frac{1}{2}\left[\frac{2-(e^{j\omega_0}+e^{-j\omega_0})Z^{-1}}{(1-e^{j\omega_0}Z^{-1})(1-e^{-j\omega_0}Z^{-1})}\right]$$

$1-e^{-j\omega_0}Z^{-1}-e^{j\omega_0}Z^{-1}$

$+e^{j\omega_0}\cdot Z^{-1}\cdot e^{-j\omega_0}\cdot Z^{-1} = Z^{-2}$

$$X(Z) = \frac{1-Z^{-1}\cos\omega_0}{1-2Z^{-1}\cos\omega_0+Z^{-2}} = \frac{1-\frac{1}{Z}\cos\omega_0}{1-\frac{2}{Z}\cos\omega_0+\frac{1}{Z^2}}$$

$$X(Z) = \frac{Z^2 - Z\cos\omega_0}{Z^2-2Z\cos\omega_0+1} \Rightarrow \frac{Z(Z-\cos\omega_0)}{Z^2-2Z\cos\omega_0+1} \quad \text{for } |Z|>1$$

(b) $\Rightarrow$ $x(n) = (\sin\omega_0 n)\,u(n) = \frac{1}{2j}\left[e^{j\omega_0 n}u(n) - e^{-j\omega_0 n}u(n)\right]$

$$X(Z) = \frac{1}{2j}\left(\frac{1}{1-e^{j\omega_0}Z^{-1}} - \frac{1}{1-e^{-j\omega_0}Z^{-1}}\right) \quad \text{ROC } |Z|>1$$

Similarly as above eg 1.

$$(\sin\omega_0 n)\,u(n) \xleftrightarrow{Z} \frac{Z^{-1}\sin\omega_0}{1-2Z^{-1}\cos\omega_0+Z^{-2}} \quad \text{ROC } |Z|>1$$

Example 4.31 Find the Z transform $u(n-5)$ using Shifting property

Soln $x_1(n) = u(n)$

$$X_1(Z) = \frac{1}{1-Z^{-1}}$$

Now apply shifting property to $x(n)$

$x(n) = u(n-5)$ [delay by 5 Samples]

$$X(Z) = Z^{-5} \frac{1}{1-Z^{-1}}$$

Example 4.32 ⇒ Find out the Z Transform of $x(n)$ using shifting property

$$X(Z) = \frac{1+\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}}$$

<u>Soln</u> → $X(Z) = \dfrac{1+\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}}$

$$= \frac{1}{1-\frac{1}{2}Z^{-1}} + \frac{\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}}$$

$x(n) = Z^{-1} X(Z)$

$$= Z^{-1}\left[\frac{1}{1-\frac{1}{2}Z^{-1}} + \frac{\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}}\right]$$

$$x(n) = Z^{-1}\left[\frac{1}{1-\frac{1}{2}Z^{-1}}\right] + Z^{-1}\left[\frac{\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}}\right]$$

$$= Z^{-1}[X_1(Z)] + Z^{-1}[X_2(Z)]$$

$$X_1(Z) = \frac{1}{1-\frac{1}{2}Z^{-1}} = Z\left\{\left(\frac{1}{2}\right)^n u(n)\right]$$

$$X_2(Z) = \frac{\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}} = \frac{1}{2}Z^{-1}\,X_1(Z)$$

apply time shifting property

$$x(n) \longleftrightarrow X(Z)$$

$$x(n-n_0) \longleftrightarrow Z^{-n_0}\,X(Z)$$

$$X_2(Z) = \frac{1}{2}Z\left[\left(\frac{1}{2}\right)^{n-1} u(n-1)\right]$$ [Here delay is of 1 unit]

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$x_2(n) = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Now apply linearity property

$$x(n) = x_1(n) + x_2(n) = \left(\frac{1}{2}\right)^n [u(n) + u(n-1)]$$ Ans

Example 4.33 Find the Z transform of

$\{\cos(\frac{n\pi}{6} + \gamma)\}$ $n \geq 0$ using time shifting property

Soln

$$\cos\omega_0 n\, u(n) \xleftrightarrow{Z} \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

Consider the term $\cos(n\frac{\pi}{6})$ $\omega_0 = \frac{\pi}{6}$

$$\cos(n\frac{\pi}{6}) \xleftrightarrow{Z} \frac{Z^2 - Z\cos(\frac{\pi}{6})}{Z^2 - 2Z\cos(\frac{\pi}{6}) + 1} = \frac{Z^2 - 0.866Z}{Z^2 - 1.732Z + 1}$$

According to time shifting property

$$x(n+k) \xleftrightarrow{Z} Z^k X(Z)$$

$$\cos(n\frac{\pi}{6} + \gamma) \xleftrightarrow{Z} Z^\gamma\left[\frac{Z^2 - 0.866Z}{Z^2 - 1.732 + 1}\right]$$

Example 4.34 Determine the Z transform of

$x(n) = 2^n u(n-2)$ using scaling property

Soln

$$u(n) \xleftrightarrow{Z} \frac{1}{1 - Z^{-1}}$$

Now consider $u(n-2) \overset{z}{\longleftrightarrow} \dfrac{z^{-2}}{1-z^{-1}}$

$a^n x(n)$

$X\left(\frac{z}{a}\right)$

$\frac{z^{-2}}{2}$

finally $2^n u(n-2) \Rightarrow \dfrac{(2^{-1}z)^{-2}}{1-(2^{-1}z)^{-1}}$

$$= \frac{4z^{-2}}{1-2z^{-1}} \quad \text{Ans}$$

Example 4.35 using scaling property determine

1. $\gamma^n \cos \omega_0 n$
2. $\gamma^n \sin \omega_0 n$

Soln → $\cos \omega_0 n \overset{z}{\longleftrightarrow} \dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}$

$$\gamma^n \cos \omega_0 n \overset{z}{\longleftrightarrow} \frac{1-(\gamma^{-1}z)^{-1}\cos\omega_0}{1-2(\gamma^{-1}z)^{-1}\cos\omega_0 + (\gamma^{-1}z)^{-2}}$$

$$= \frac{1-\gamma z^{-1}\cos\omega_0}{1-2\gamma z^{-1}\cos\omega_0 + \gamma^2 z^{-2}} \quad \text{Ans}$$

2. Similarly we obtain

$$Z[\gamma^n \sin \omega_0 n] = \frac{\gamma Z^{-1} \sin \omega_0}{1-2\gamma Z^{-1}\cos\omega_0 + \gamma^2 Z^{-2}}$$

$$= \frac{\gamma Z \sin \omega_0}{Z^2 - 2\gamma Z \cos\omega_0 + \gamma^2}$$

Example 4.36 → Compute the Convolution $x(n)$ of the signals.

$$x_1(n) = \{\underset{\uparrow}{4}, -2, 1\}$$

$$x_2(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Soln

$$x_1(n) = \{\underset{\uparrow}{4}, -2, 1\}$$

$$X_1(Z) = 4 - 2Z^{-1} + 1Z^{-2}$$

$$x_2(n) = \{0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1, 1, 0, 0\}$$

$$X_2(Z) = 1 + Z^{-1} + Z^{-2} + Z^{-3} + Z^{-4} + Z^{-5}$$

Now $x_1(n) \circledast x_2(n) \xleftrightarrow{Z} X_1(Z)\, X_2(Z)$

Again using differentiation property

$Y(Z) = Z$ transform of $n^2 u(n) = n \times n u(n)$

$$= -Z \frac{d}{dZ}\left[\frac{Z^{-1}}{(1-Z^{-1})^2}\right] = \frac{Z^{-1}(1+Z^{-1})}{(1-Z^{-1})^3} \quad \text{Ans}$$

Example 4.38 Determine initial & final value of $x(n)$.

$$X(Z) = 2+3Z^{-1}+4Z^{-2}$$

Soln Applying initial value

$$x(0) = \lim_{n\to 0} x(n) = \lim_{|Z|\to\infty} X(Z)$$

$$X(Z) = 2+3Z^{-1}+4Z^{-2}$$

$$x(0) = \lim_{Z\to\infty} X(Z) = 2+\frac{3}{\infty}+\frac{4}{\infty} = 2+0+0$$

$$x(0) = 2$$

Applying final value theorem

$$x(\infty) = \lim_{|Z|\to 1} (1-Z^{-1})[X(Z)]$$

$$\lim_{|Z|\to 1}\left[(1-Z^{-1})(2+3Z^{-1}+4Z^{-2})\right]$$

$$= \lim_{|Z|\to 1}\left[2+Z^{-1}+Z^{-2}-4Z^{-3}\right]$$

$$= \lim_{|Z|\to 1}\left[2+\frac{1}{1}+\frac{1}{1}-\frac{4}{1}\right]$$

$$= 2+1+1-4 = 0$$

$x(\infty) = 0$ Ans.

Example 4.39 determine the Z transform of $x(n) = (2)^{n+3}\, u(n-1)$ using scaling property also draw ROC

Soln

$$x(n) = (2)^{n+3}\, u(n-1)$$

$$x(n) = 2^n \cdot 2^3\, u(n-1)$$

$$x(n) = 8 \cdot 2^n\, u(n-1) \quad \text{--- (1)}$$

Z transform of $u(n-1)$

$$u(n) = \frac{Z}{Z-1} \quad \text{ROC } |Z|>1$$

using shifting property

$$u(n-1) = Z^{-1}\left[\frac{Z}{Z-1}\right] \quad \text{ROC } |Z|>1$$

$$Z\{u(n-1)\} = \frac{1}{Z-1} \quad |Z|>1$$

According to scaling property

$$Z\{a^n u(n)\} = X\left(\frac{Z}{a}\right)$$

$a=2$ applying scaling property

$$Z\{2^n u(n-1)\} = \frac{1}{\frac{Z}{2}-1} \quad \text{ROC } \left|\frac{Z}{2}\right|>1$$

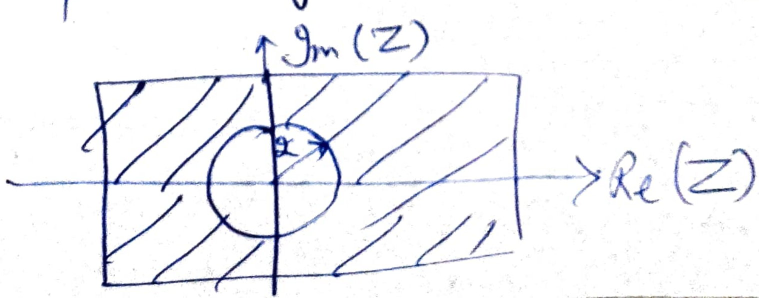
$$Z\{2^n u(n-1)\} = \frac{2}{Z-2} \quad \text{ROC } |Z|>2$$

From eqⁿ (1)

$$Z\{8\, 2^n u(n-1)) = 8\cdot\frac{2}{Z-2} \quad \text{ROC } |Z|>2$$

$$X(Z) = \frac{16}{Z-2} \quad \text{ROC } |Z|>2$$

Exterior part of circle having radius 2



Example 4.40 Find the Z transform of following sequence

(i) $x(n) = a^n u(n-1)$

(ii) $= -a^n u(n-1)$

Soln $a^n u(n-1)$

$$Z(u(n)) = \frac{z}{z-1} \quad \text{ROC } |z| > 1$$

using shifting property

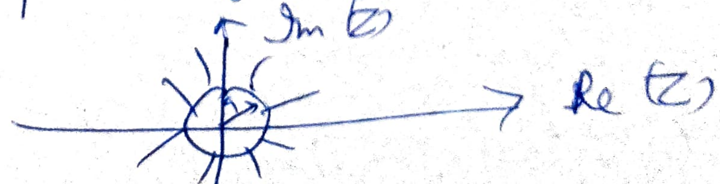
$$Z(u(n-1)) = z^{-1} \frac{z}{z-1} = \frac{1}{z-1} \quad \text{ROC } |z| > 1$$

$Z[a^n u(n-1)$ Apply scaling property

$$Z\{a^n u(n-1) = \frac{1}{\frac{z}{a} - 1} \quad \text{ROC } \left|\frac{z}{a}\right| > 1$$

$$Z\{a^n u(n-1) = \frac{a}{z-a} \quad \text{ROC } |z| > a$$

Exterior part of circle having radius a



(2) $x(n) = -a^n u(n-1)$

$$Z\{a^n u(n-1)\} = \frac{a}{z-a} \quad ROC\ |z| > a$$

Thus

$$Z\{-a^n u(n-1)\} = -\frac{a}{z-a} \quad |z| > a$$

$$Z\{-a^n u(n-1)\} = \frac{a}{a-z} \quad |z| > a$$